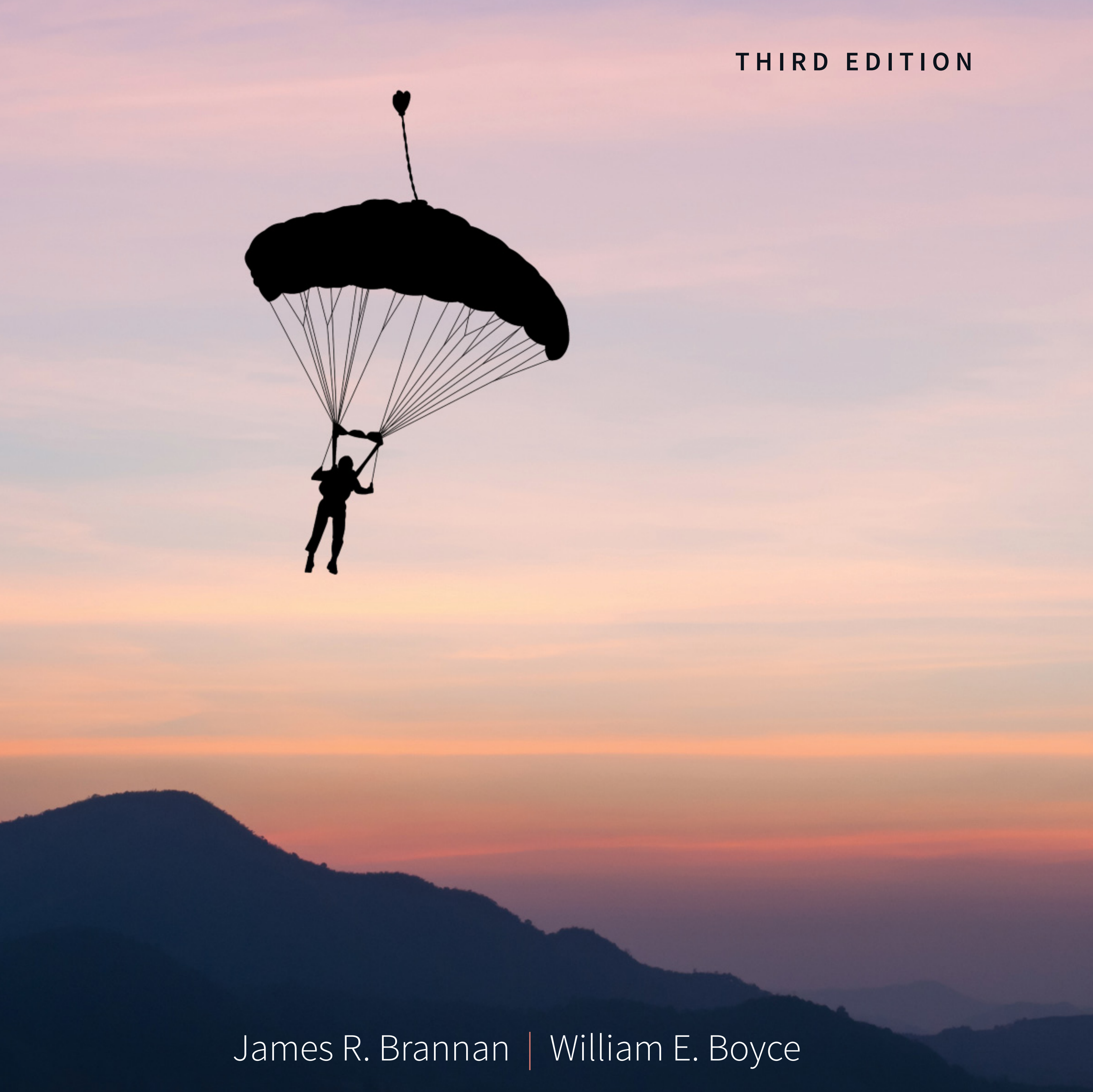
THIRD EDITION



James R. Brannan | William E. Boyce

DIFFERENTIAL EQUATIONS

An Introduction to Modern Methods and Applications

WILEY

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ISBN 978-1-118-53177-8
ISBN 978-1-118-98122-1 (BRV)

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

PREFACE

This is a textbook for a first course in differential equations. The book is intended for science and engineering majors who have completed the calculus sequence, but not necessarily a first course in linear algebra. It emphasizes a systems approach to the subject and integrates the use of modern computing technology in the context of contemporary applications from engineering and science.

Our goal in writing this text is to provide these students with both an introduction to, and a survey of, modern methods, applications, and theory of differential equations that is likely to serve them well in their chosen field of study. The subject matter is presented in a manner consistent with the way practitioners use differential equations in their work; technology is used freely, with more emphasis on methods, modeling, graphical representation, qualitative concepts, and geometric intuition than on theory.

Notable Changes in the Third Edition

This edition is a substantial revision of the second edition. The most significant changes are:

- **Enhanced Page Layout** We have placed important results, theorems, definitions, and tables in highlighted boxes and have put subheadings just before the most important topics in each section. This should enhance readability for both students and instructors and help students to review material for exams.
- **Increased Emphasis on Qualitative Methods** Qualitative methods are introduced early. Throughout the text, new examples and problems have been added that require the student to use qualitative methods to analyze solution behavior and dependence of solutions on parameters.
- **New Chapter on Numerical Methods** Discussions on numerical methods, dispersed over three chapters in the second edition, have been revised and reassembled as a unit in Chapter 8. However, the first three sections of Chapter 8 can be studied by students after they have studied Chapter 1 and the first two sections of Chapter 2.
- **Chapter 1: Introduction** This chapter has been reduced to three sections. In Section 1.1 we follow up on introductory models and concepts with a discussion of the art and craft of mathematical modeling. Section 1.2 has been replaced by an early introduction to qualitative methods, in particular, phase lines and direction fields. Linearization and stability properties of equilibrium solutions are also discussed. In Section 1.3 we cover definitions, classification, and terminology to help give the student an organizational overview of the subject of differential equations.
- **Chapter 2: First Order Differential Equations** New mathematical modeling problems have been added to Section 2.3, and a new Section 2.7 on subsitution methods has been added. Sections on numerical methods have been moved to Chapter 8.
- **Chapter 3: Systems of Two First Order Equations** The discussion of Wronskians and fundamental sets of solutions has been supplemented with the definition of, and relationship to, linearly independent solutions of two-dimensional linear systems.
- **Chapter 4: Second Order Linear Equations** Section 4.6 on forced vibrations, frequency response, and resonance has been rewritten to improve its readability for students and instructors.

- **Chapter 10: Orthogonal Functions, Fourier Series and Boundary-Value Problems** This chapter gives a unified treatment of classical and generalized Fourier series in the framework of orthogonal families in the space PC[a, b].
- **Chapter 11: Elementary Partial Differential Equations** Material and projects on the heat equation, wave equation, and Laplace's equation that appeared in Chapters 9 and 10 of the second edition, have been moved to Chapter 11 in the third edition.
- **Miscellaneous Changes and Additions** Changes have been made in current problems, and new problems have been added to many of the section problem sets. For ease in assigning homework, boldface headings have been added to partition the problems into groups corresponding to major topics discussed in the section.

Major Features

- **Flexible Organization.** Chapters are arranged, and sections and projects are structured, to facilitate choosing from a variety of possible course configurations depending on desired course goals, topics, and depth of coverage.
- **Numerous and Varied Problems.** Throughout the text, section exercises of varying levels of difficulty give students hands-on experience in modeling, analysis, and computer experimentation.
- **Emphasis on Systems.** Systems of first order equations, a central and unifying theme of the text, are introduced early, in Chapter 3, and are used frequently thereafter.
- **Linear Algebra and Matrix Methods.** Two-dimensional linear algebra sufficient for the study of two first order equations, taken up in Chapter 3, is presented in Section 3.1. Linear algebra and matrix methods required for the study of linear systems of dimension n (Chapter 6) are treated in Appendix A.
- **Optional Computing Exercises.** In most cases, problems requesting computer-generated solutions and graphics are optional.
- **Visual Elements.** The text contains a large number of illustrations and graphs. In addition, many of the problems ask the student to compute and plot solutions of differential equations.
- **Contemporary Project Applications.** Optional projects at the end of all but one of Chapters 2 through 11 integrate subject matter in the context of exciting, often contemporary, applications in science and engineering.
- **Laplace Transforms.** A detailed chapter on Laplace transforms discusses systems, discontinuous and impulsive input functions, transfer functions, feedback control systems, poles, and stability.
- **Control Theory.** Ideas and methods from the important application area of control theory are introduced in some examples, some projects, and in the last section on Laplace transforms. All this material is optional.
- **Recurring Themes and Applications.** Important themes, methods, and applications, such as dynamical system formulation, phase portraits, linearization, stability of equilibrium solutions, vibrating systems, and frequency response, are revisited and reexamined in a variety of mathematical models under different mathematical settings.
- **Chapter Summaries.** A summary at the end of each chapter provides students and instructors with a bird's-eye view of the most important ideas in the chapter.
- **Answers to Problems.** Answers to selected odd-numbered problems are provided at the end of the book; many of them are accompanied by a figure.

Problems that require the use of a computer are marked with . While we feel that students will benefit from using the computer on those problems where numerical approximations or computer-generated graphics are requested, in most problems it is clear that use of a computer, or even a graphing calculator, is optional. Furthermore there are a large number of problems that do not require the use of a computer. Thus the book can easily be used in a course without using any technology.

Relation of This Text to Boyce and DiPrima

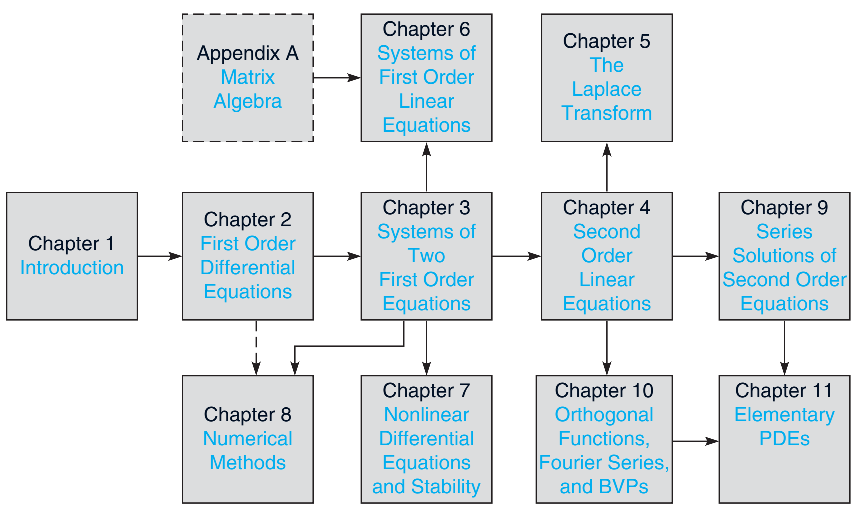
Brannan and Boyce is an offshoot of the well-known textbook by Boyce and DiPrima. Readers familiar with Boyce and DiPrima will doubtless recognize in the present book some of the hallmark features that distinguish that textbook.

To help avoid confusion among potential users of either text, the primary differences are described below:

- Brannan and Boyce is more sharply focused on the needs of students of engineering and science, whereas Boyce and DiPrima targets a somewhat more general audience, including engineers and scientists.
- Brannan and Boyce is intended to be more consistent with the way contemporary scientists and engineers actually use differential equations in the workplace.
- Brannan and Boyce emphasizes systems of first order equations, introducing them earlier, and also examining them in more detail than Boyce and DiPrima. Brannan and Boyce has an extensive appendix on matrix algebra to support the treatment of systems in n dimensions.
- Brannan and Boyce integrates the use of computers more thoroughly than Boyce and DiPrima, and assumes that most students will use computers to generate approximate solutions and graphs throughout the book.
- Brannan and Boyce emphasizes contemporary applications to a greater extent than Boyce and DiPrima, primarily through end-of-chapter projects.
- Brannan and Boyce makes somewhat more use of graphs, with more emphasis on phase plane displays, and uses engineering language (e.g., state variables, transfer functions, gain functions, and poles) to a greater extent than Boyce and DiPrima.

Options for Course Structure

Chapter dependencies are shown in the following block diagram:



The book has much built-in flexibility and allows instructors to choose from many options. Depending on the course goals of the instructor and background of the students, selected sections may be covered lightly or even omitted.

- Chapters 5, 6, and 7 are independent of each other, and Chapters 6 and 7 are also independent of Chapter 4. It is possible to spend much class time on one of these chapters, or class time can be spread over two or more of them.
- The amount of time devoted to projects is entirely up to the instructor.
- For an honors class, a class consisting of students who have already had a course in linear algebra, or a course in which linear algebra is to be emphasized, Chapter 6 may be taken up immediately following Chapter 2. In this case, material from Appendix A, as well as sections, examples, and problems from Chapters 3 and 4, may be selected as needed or desired. This offers the possibility of spending more class time on Chapters 5, 7, and/or selected projects.

Acknowledgments

It is a pleasure to offer our grateful appreciation to the many people who have generously assisted in the preparation of this book.

To the individuals listed below who reviewed parts or all of the third edition manuscript at various stages of its development:

Miklós Bóna, University of Florida
Mark W. Brittenham, University of Nebraska
Yanzhao Cao, Auburn University
Doug Cenzer, University of Florida
Leonard Chastkofsky, University of Georgia
Jon M. Collis, Colorado School of Mines
Domenico D'Alessandro, Iowa State University
Shaozhong Deng, University of North Carolina at Charlotte
Patricia J. Diute, Rochester Institute of Technology
Behzad Djafari Rouhani, University of Texas at El Paso
Alina N. Duca, North Carolina State University
Marek Z. Elżanowski, Portland State University
Vincent Graziano, Case Western Reserve University
Mansoor A. Haider, North Carolina State University
M. D. Hendon, University of Georgia
Mohamed A Khamsi, University of Texas at El Paso
Marcus A. Khuri, Stony Brook University
Richard C. Le Borne, Tennessee Technological University
Glenn Ledder, University of Nebraska-Lincoln
Kristopher Lee, Iowa State University
Jens Lorenz, University of New Mexico
Aldo J. Manfroi, University of Illinois
Marcus McGuff, Austin Community College

William F. Moss, Clemson University

Mike Nicholas, Colorado School of Mines

Mohamed Ait Nouh, University of Texas at El Paso

Francis J. Poulin, University of Waterloo

Mary Jarratt Smith, Boise State University

Stephen J. Summers, University of Florida

Yi Sun, University of South Carolina Kyle Thompson, North Carolina State University

Stella Thistlethwaite, University of Tennessee, Knoxville

Vincent Vatter, University of Florida

Pu Patrick Wang, University of Alabama

Dongming Wei, Nazarbayev University

Larissa Williamson, University of Florida

Hafizah Yahya, University of Alberta

Konstantin Zuev, University of Southern California.

To Mark McKibben, West Chester University; Doug Meade, University of South Carolina; Bill Siegmann, Rensselaer Polytechnic Institute, for their contributions to the revision.

To Jennifer Blue, SUNY Empire State College, for accuracy checking page proofs.

To the editorial and production staff of John Wiley and Sons, Inc., identified on page iv, who were responsible for turning our manuscript into a finished book. In the process, they maintained the highest standards of professionalism.

We also wish to acknowledge the less tangible contributions of our friend and colleague, the late Richard DiPrima. Parts of this book draw extensively on the book on differential equations by Boyce and DiPrima. Since Dick DiPrima was an equal partner in creating the early editions of that book, his influence lives on more than thirty years after his untimely death.

Finally, and most important of all, we thank our wives Cheryl and Elsa for their understanding, encouragement, and patience throughout the writing and production of this book. Without their support it would have been much more difficult, if not impossible, for us to complete this project.

James R. Brannan
Clemson, South Carolina

William E. Boyce
Latham, New York

Supplemental Resources for Instructors and Students

An Instructor's Solutions Manual, includes solutions for all problems in the text.

A Student Solutions Manual, ISBN 9781118981252, includes solutions for selected problems in the text.

A Companion website, www.wiley.com/college/brannan, provides a wealth of resources for students and instructors, including:

- PowerPoint slides of important ideas and graphics for study and note taking.
- Online Only Projects—these projects are like the end-of-chapter projects in the text. They present contemporary problems that are not usually included among traditional differential equations topics. Many of the projects involve applications derived from a variety of disciplines and integrate or extend theories and methods presented in core material.
- Mathematica, Maple, and MATLAB data files are provided for selected end-of-section or end-of-chapter problems in the text allowing for further exploration of important ideas in the course utilizing these computer algebra and numerical analysis packages. Students will benefit from using the computer on problems where numerical approximations or computer generated graphics are requested.
- Review of Integration—An online review of integration techniques is provided for students who need a refresher.

WileyPLUS: Expect More from Your Classroom Technology

This text is supported by *WileyPLUS*—a powerful and highly integrated suite of teaching and learning resources designed to bridge the gap between what happens in the classroom and what happens at home. *WileyPLUS* includes a complete online version of the text, algorithmically generated exercises, all of the text supplements, plus course and homework management tools, in one easy-to-use website.

Organized around the everyday activities you perform in class, *WileyPLUS* helps you:

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- Administer Your Course: You can easily integrate *WileyPLUS* with another course management system, gradebooks, or other resources you are using in your class, enabling you to build your course, your way.

CONTENTS

CHAPTER ONE


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Introduction

In this introductory chapter we formulate several problems that illustrate basic ideas that reoccur frequently in this book.

In Section 1.1 we discuss two mathematical models, one from physics and one from population biology. Each mathematical model is a differential equation—an equation involving the rate of change of a variable with respect to time. Using these models as examples, we introduce some basic terminology, explore the notion of a solution of a differential equation, and end with an overview of the art and craft of mathematical modeling.

It is not always possible to find analytic, closed-form solutions of a differential equation. In Section 1.2 we look at two graphical methods for studying the qualitative behavior of solutions: phase lines and direction fields. Although we will learn how to sketch direction fields by hand, we will use the computer to draw them.

Sections 1.1 and 1.2 give us a glimpse of two of the three major methods of studying differential equations, the **analytical** method and the **geometric** method, respectively. We defer study of the third major method—**numerical**—to Chapter 8. However, you may study the first three sections of Chapter 8 immediately after Chapter 1.

In Section 1.3 we present some important definitions and commonly used terminology in conjunction with different ways of classifying differential equations. Classification schemes provide organizational structure for the book and help give you perspective on the subject of differential equations.

1.1 Mathematical Models and Solutions

Many of the principles, or laws, underlying the behavior of the natural world are statements, or relations, involving rates in which one variable, say, y, changes with respect to another variable, t, for example. Most often, these relations take the form of equations containing y and certain of the derivatives $y', y'', \ldots, y^{(n)}$ of y with respect to t. The resulting equations are then referred to as **differential equations**. Some examples of differential equations that will be studied in detail later on in the text, are:

$y' = r\left(1 - \dfrac{y}{K}\right)y,$	an equation for population dynamics,
$my'' + \gamma y' + ky = 0,$	the equation for a damped spring-mass system, and
$\theta'' + \dfrac{g}{l}\sin(\theta) = 0,$	the pendulum equation.

The subject of differential equations was motivated by problems in mechanics, elasticity, astronomy, and geometry during the latter part of the 17th century. Inventions (or discoveries) in theory, methods, and notation evolved concurrently with innovations in calculus. Since their early historical origins, the number and variety of problems to which differential equations are applied have grown substantially. Today, scientists and engineers use differential equations to study problems in all fields of science and engineering, as well as in several of the business and social sciences. Some representative problems from these fields are shown below.

Applications of Differential Equations

- airplane and ship design
- earthquake detection and prediction
- controlling the flight of ships and rockets
- modeling the dynamic behavior of nerve cells
- describing the behavior of economic systems
- heat transfer
- wave propagation
- weather forecasting
- designing medical imaging technologies
- determining the price of financial derivatives
- forecasting and managing the harvesting of fish populations
- designing optimal vaccination policies to prevent the spread of disease

The common thread that links these applications is that they all deal with systems that evolve in time. Differential equations is the mathematical apparatus that we use to study such systems.

We often refer to a differential equation that describes some physical process as a **mathematical model** of the process; many such models are discussed throughout this book. In this section we construct a model from physics and a model from population biology. Each model results in an equation that can be solved by using an integration technique from calculus. These examples suggest that even simple differential equations can provide useful models of important physical systems.

Heat Transfer: Newton's Law of Cooling

EXAMPLE 1

If a material object is hotter or colder than the surrounding environment, its temperature will approach the temperature of the environment. If the object is warmer than the environment, its temperature will decrease. If the object is cooler than the environment, its temperature will increase. Sir Isaac Newton postulated that the rate of change of the temperature of the object is negatively proportional to the difference between its temperature and the temperature of the surroundings (the **ambient temperature**). This principle is referred to as **Newton's law of cooling**.

Suppose we let $u(t)$ denote the temperature of the object at time t, and let T be the ambient temperature (see Figure 1.1.1). Then du/dt is the rate at which the temperature of the object changes. From Newton, we know that du/dt is proportional to $-(u - T)$. Introducing a positive constant of proportionality k called the **transmission coefficient**, we then get the differential equation

$$\frac{du}{dt} = -k(u - T), \qquad \text{or} \qquad u' = -k(u - T). \tag{1}$$

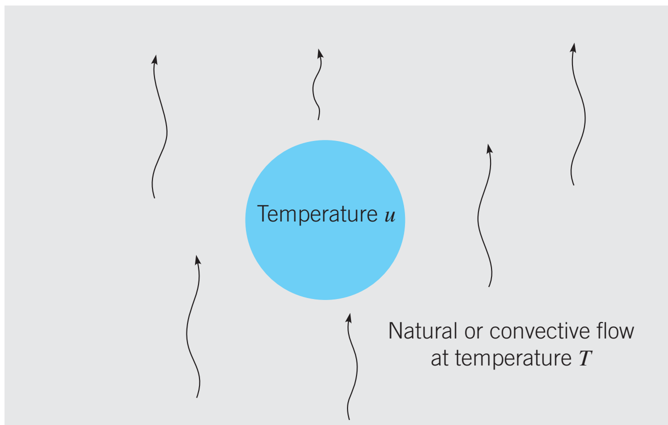


FIGURE 1.1.1 Newton's Law of Cooling: The time rate of change of u, du/dt, is negatively proportional to $u - T$: $du/dt \propto -(u - T)$.

Note that the minus sign on the right side of Eq. (1) causes du/dt to be negative if $u(t) > T$, while du/dt is positive if $u(t) < T$. The transmission coefficient measures the rate of heat exchange between the object and its surroundings. If k is large, the rate of heat exchange is rapid. If k is small, the rate of heat exchange is slow. This would be the case, for example, if the object was surrounded by thick insulating material.

The temperatures u and T are measured in either degrees Fahrenheit (°F) or degrees Celsius (°C). Time is usually measured in units that are convenient for expressing time intervals over which significant changes in u occur, such as minutes, hours, or days. Since the left side of Eq. (1) has units of temperature per unit time, k must have the units of $(\text{time})^{-1}$.

Newton's law of cooling is applicable to situations in which the temperature of the object is approximately uniform at all times. This is the case for small objects that conduct heat easily, or containers filled with a fluid that is well mixed. Thus, we expect the model to be reasonably accurate in predicting the temperature of a small copper sphere, a well-stirred cup of coffee, or a house in which the air is continuously circulated, but the model would not be very accurate for predicting the temperature of a roast in an oven.

Terminology

Let us assume that the ambient temperature T in Eq. (1) is a constant, say, $T = T_0$, so that Eq. (1) becomes

$$u' = -k(u - T_0). \tag{2}$$

In Section 1.2 we consider an example in which T depends on t. Common mathematical terminology for the quantities that appear in this equation are:

time	t	is an **independent variable**,
temperature	u	is a **dependent variable** because it depends on t,
	k and T_0	are **parameters** in the model.

The equation is an **ordinary differential equation** because it has one, and only one, independent variable. Consequently, the derivative in Eq. (2) is an ordinary derivative. It is a **first order** equation because the highest order derivative that appears in the equation is the first derivative. The dependency of u on t implies that u is, in fact, a function of t, say, $u = \phi(t)$. Thus when we write Eq. (2), three questions may, after a bit of reflection, come to mind:

1. "Is there actually a function $u = \phi(t)$, with derivative $u' = d\phi/dt$, that makes Eq. (2) a true statement for each time t?" If such a function exists, it is called a solution of the differential equation.
2. "If the differential equation does have a solution, how can we find it?"
3. "What can we do with this solution, once we have found it?"

In addition to methods used to derive mathematical models, answers to these types of questions are the main subjects of inquiry in this book.

Solutions and Integral Curves

By a **solution** of Eq. (2), we mean a differentiable function $u = \phi(t)$ that satisfies the equation. One solution of Eq. (2) is $u = T_0$, since Eq. (2) reduces to the identity $0 = 0$ when T_0 is substituted for u in the equation. In other words, "It works when we put it into the equation." The constant solution $u = T_0$ is referred to as an **equilibrium solution** of Eq. (2). Although simple, equilibrium solutions usually play an important role in understanding the behavior of other solutions. In Section 1.2 we will consider them in a more general setting.

If we assume that $u \neq T_0$, we can discover other solutions of Eq. (2) by first rewriting it in the form

$$\frac{du/dt}{u - T_0} = -k. \tag{3}$$

By the chain rule the left side of Eq. (3) is the derivative of $\ln|u - T_0|$ with respect to t, so we have

$$\frac{d}{dt}\ln|u - T_0| = -k. \tag{4}$$

Then, by integrating both sides of Eq. (4), we obtain

$$\ln|u - T_0| = -kt + C, \tag{5}$$

where C is an arbitrary constant of integration. Therefore, by taking the exponential of both sides of Eq. (5), we find that

$$|u - T_0| = e^{-kt+C} = e^C e^{-kt}, \tag{6}$$

or

$$u - T_0 = \pm e^C e^{-kt}. \tag{7}$$

Thus

$$u = T_0 + ce^{-kt} \tag{8}$$

is a solution of Eq. (2), where $c = \pm e^C$ is also an arbitrary (nonzero) constant. Note that if we allow c to take the value zero, then the constant solution $u = T_0$ is also contained in the expression (8). The expression (8) contains all possible solutions of Eq. (2) and is called the **general solution** of the equation.

Given a differential equation, the usual problem is to find solutions of the equation. However, it is also important to be able to determine whether a particular function is a solution of the equation. Thus, if we were simply asked to verify that u in Eq. (8) is a solution of Eq. (2), then we would need to substitute $T_0 + ce^{-kt}$ for u in Eq. (2) and show that the equation reduces to an identity, as we now demonstrate.

EXAMPLE 2

Verify by substitution that $u = T_0 + ce^{-kt}$, where c is an arbitrary real number, is a solution of Eq. (2),

$$u' = -k(u - T_0), \tag{9}$$

on the interval $-\infty < t < \infty$.

Substituting $\phi(t) = T_0 + ce^{-kt}$ for u in the left side of the equation gives $\phi'(t) = -kce^{-kt}$ while substituting $\phi(t)$ for u into the right side yields $-k(T_0 + ce^{-kt} - T_0) = -kce^{-kt}$. Thus, upon substitution, Eq. (2) reduces to the identity

$$\underbrace{-kce^{-kt}}_{\phi'(t)} = \underbrace{-kce^{-kt}}_{-k(\phi(t)-T_0)}, \qquad -\infty < t < \infty,$$

for each real number c and each value of the parameter k.

Integral Curves. The geometrical representation of the general solution (8) is an infinite family of curves in the tu-plane called **integral curves**. Each integral curve is associated with a particular value of c; it is the graph of the solution corresponding to that value of c.

Although we can sketch, by hand, qualitatively correct integral curves described by Eq. (8), we will assign numerical values to k and T_0, and then use a computer to plot the graph of Eq. (8) for some different values of c. Setting $k = 1.5$ day^{-1} and $T_0 = 60$°F in Eq. (2) and Eq. (8) gives us

$$\frac{du}{dt} = -1.5(u - 60), \tag{10}$$

with the corresponding general solution

$$u = 60 + ce^{-1.5t}. \tag{11}$$

In Figure 1.1.2 we show several integral curves of Eq. (10) obtained by plotting the graph of the function in Eq. (11) for different values of c. Note that all solutions approach the equilibrium solution $u = 60$ as $t \to \infty$.

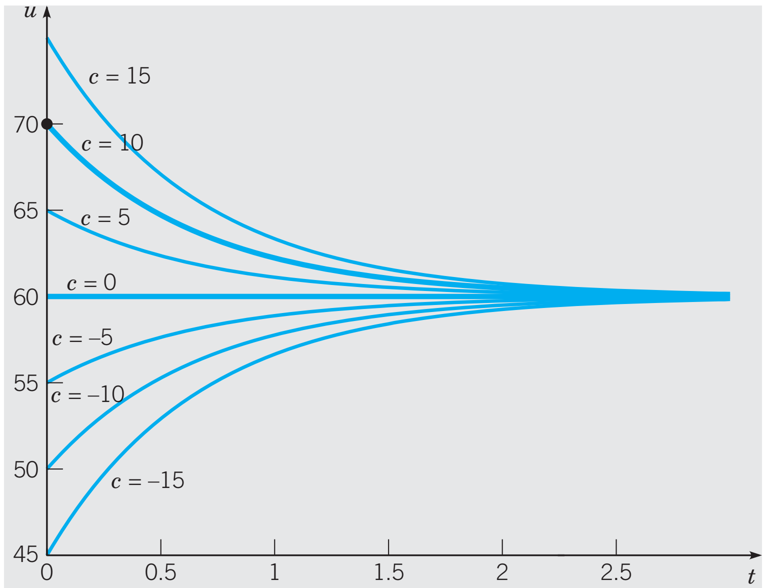


FIGURE 1.1.2 Integral curves of $u' = -1.5(u - 60)$. The curve corresponding to $c = 10$ in Eq. (11) is the graph of $u = 60 + 10e^{-1.5t}$, the solution satisfying the initial condition $u(0) = 70$. The curve corresponding to $c = 0$ in Eq. (11) is the graph of the equilibrium solution $u = 60$, which satisfies the initial condition $u(0) = 60$.

Initial Value Problems

Frequently, we want to focus our attention on a single member of the infinite family of solutions by specifying the value of the arbitrary constant. Most often, we do this by specifying a point that must lie on the graph of the solution. For example, to determine the constant c in Eq. (11), we could require that the temperature have a given value at a certain time, such as the value 70 at time $t = 0$. In other words, the graph of the solution must pass through the point $(0, 70)$. Symbolically, we can express this condition as

$$u(0) = 70. \tag{12}$$

Then, substituting $t = 0$ and $u = 70$ into Eq. (11), we obtain

$$70 = 60 + c.$$

Hence $c = 10$, and by inserting this value in Eq. (11), we obtain the desired solution, namely,

$$u = 60 + 10e^{-1.5t}. \tag{13}$$

The graph of the solution (13) is the thick curve, labeled by $c = 10$, in Figure 1.1.2. The additional condition (12) that we used to determine c is an example of an **initial condition**.

The differential equation (10) together with the initial condition (12) form an **initial value problem**.

Note that the solution of Eq. (10) subject to the initial condition $u(0) = 60$ is the equilibrium solution $u = 60$, the thick curve labeled by $c = 0$ in Figure 1.1.2.

Population Biology

Next we consider a problem in population biology. To help control the field mouse population in his orchards, in an economical and ecofriendly way, a fruit farmer installs nesting boxes for barn owls, predators for whom mice are a natural food supply. In the absence of predators we assume that the rate of change of the mouse population is proportional to the current population; for example, if the population doubles, then the number of births per unit time also doubles. This assumption is not a well-established physical law (such as the laws of thermodynamics, which underlie Newton's law of cooling in Example 1), but it is a common initial hypothesis[1] in a study of population growth. If we denote time by t and the mouse population by $p(t)$, then the assumption about population growth can be expressed by the equation

$$\frac{dp}{dt} = rp, \tag{14}$$

where the proportionality factor r is called the **rate constant** or **growth rate**.

As a simple model for the effect of the owl population on the mouse population, let us assume that the owls consume the mice at a constant predation rate a. By modifying Eq. (14) to take this into account, we obtain the equation

$$\frac{dp}{dt} = rp - a, \tag{15}$$

where both r and a are positive. Thus the rate of change of the mouse population, dp/dt, is the net effect of the growth term rp and the predation term $-a$. Depending on the values of p, r, and a, the value of dp/dt may be of either sign.

EXAMPLE 3

Suppose that the growth rate for the field mice is 0.5/month and that the owls kill 15 mice per day. Determine appropriate values for the parameters in Eq. (15), find the general solution of the resulting equation, and graph several solutions, including any equilibrium solutions.

We naturally assume that p is the number of individuals in the mouse population at time t. We can choose our units for time to be whatever seems most convenient; the two obvious possibilities are days or months. If we choose to measure time in months, then the growth term is $0.5p$ and the predation term is $-(15 \text{ mice/day}) \cdot (30 \text{ days/month}) = -450$ mice/month, assuming an average month of 30 days. Thus Eq. (15) becomes

$$\frac{dp}{dt} = 0.5p - 450, \tag{16}$$

where each term has the units of mice/month.

By following the same steps that led to the general solution of Eq. (2), we find that the general solution of Eq. (16) is

$$p = 900 + ce^{t/2}, \tag{17}$$

where c is again a constant of integration.

[1] A somewhat better model of population growth is discussed in Section 2.5.

Integral curves for Eq. (16) are shown in Figure 1.1.3. For sufficiently large values of p it can be seen from the figure, or directly from Eq. (16) itself, that dp/dt is positive, so that solutions increase. On the other hand, for small values of p the opposite is the case. Again, the critical value of p that separates solutions that increase from those that decrease is the value of p for which dp/dt is zero. By setting dp/dt equal to zero in Eq. (16) and then solving for p, we find the equilibrium solution $p = 900$ for which the growth term and the predation term in Eq. (16) are exactly balanced. This corresponds to the choice $c = 0$ in the general solution (17).

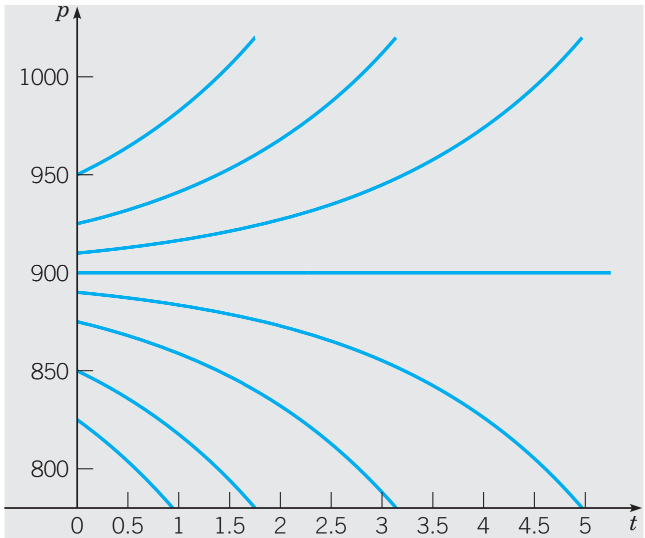


FIGURE 1.1.3 Integral curves, including the equilibrium solution $p = 900$, for $p' = 0.5p - 450$.

Solutions of the more general equation (15), in which the growth rate and the predation rate are unspecified, behave very much like those of Eq. (16). The equilibrium solution of Eq. (15) is $p = a/r$. Solutions above the equilibrium solution increase, while those below it decrease.

Constructing Mathematical Models

Mathematical modeling is the craft, and art, of using mathematics to describe and understand real-world phenomena. A viable mathematical model can be used to test ideas, make predictions, and aid in design and control problems that are associated with the phenomena. For instance, in Example 1, we constructed the differential equation

$$\frac{du}{dt} = -k(u - T) \tag{18}$$

to model heat exchange between an object and its surroundings. Recall that $u(t)$ is the time-dependent variable representing the temperature of the object and T is the temperature of the surroundings. If the value of u is known at time $t = 0$, and the values of the parameters T and k are known, solutions of this differential equation tell us what the temperature of the object will be for times $t > 0$.

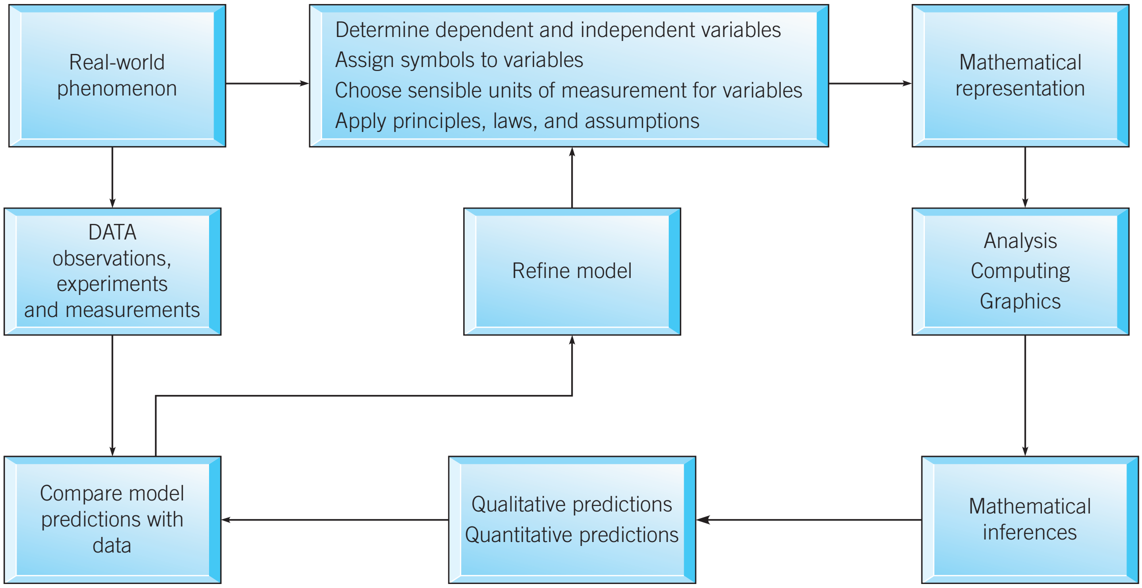


FIGURE 1.1.4 A diagram of the modeling process.

The steps used to arrive at Eq. (18) are typical of the steps used to construct any mathematical model. It is, therefore, worthwhile to illustrate the general process by a system flow diagram, as in Figure 1.1.4.

In the Problems for this section, and for many other sections of this textbook, we ask you to construct differential equation models of various real-world phenomena. In constructing mathematical models, you will find that each problem is different. Although the modeling process, in broad outline, is well represented by the above diagram, it is not a skill that can be reduced to the observance of a set of prescribed rules. Successful modeling usually requires that the modeler be intimate with the field in which the problem originates. However experience has shown that the very act of attempting to construct a mathematical model forces the modeler to ask the most cogent questions about the phenomenon being investigated:

1. What is the purpose of the model?
2. What aspects of the phenomenon are most important for the intended uses of the model?
3. What can we measure or observe?
4. What are the relevant variables; what is their relationship to the measurements?
5. Are there well-established principles (such as physical laws, or economic laws) to guide us in formulating the model?
6. In terms of the variables, how do we mathematically represent the interaction of various components of the phenomenon?
7. What simplifying assumptions can we make?
8. Do conclusions and predictions of the model agree with experiment and observations?
9. What additional experiments are suggested by the model?
10. What are limitations of the model?

For many applied mathematicians, engineers, and scientists, mathematical modeling is akin to poetry—an art form and creative act employing language that adheres to form and conventions. Likewise, there are rules (e.g., physical laws) that the mathematical modeler must follow, yet he or she has access to a myriad of mathematical tools (the language) for describing the phenomenon under investigation. History abounds with the names of scientists, mathematicians, and engineers, driven by the desire to understand nature and advance technology, who have engaged in the practice of mathematical modeling: Newton, Euler, von Kármán, Verhulst, Maxwell, Rayleigh, Navier, Stokes, Heaviside, Einstein, Schrödinger, and so on. Their contributions have literally changed the world. Nowadays, mathematical modeling is carried out in universities, government agencies and laboratories, business and industrial concerns, policy think tanks, and institutes dedicated to research and education. For many practitioners of mathematical modeling, it is, in a sense, their *raison d'être*.

PROBLEMS

1. Newton's Law of Cooling. A cup of hot coffee has a temperature of 200°F when freshly poured, and is left in a room at 70°F. One minute later the coffee has cooled to 190°F.

(a) Assume that Newton's law of cooling applies. Write down an initial value problem that models the temperature of the coffee.
(b) Determine when the coffee reaches a temperature of 170°F.

2. Blood plasma is stored at 40°F. Before it can be used, it must be at 90°F. When the plasma is placed in an oven at 120°F, it takes 45 minutes (min) for the plasma to warm to 90°F. Assume Newton's law of cooling applies. How long will it take the plasma to warm to 90°F if the oven temperature is set at 100°F?

3. At 11:09 p.m. a forensics expert arrives at a crime scene where a dead body has just been found. Immediately, she takes the temperature of the body and finds it to be 80°F. She also notes that the programmable thermostat shows that the room has been kept at a constant 68°F for the past 3 days. After evidence from the crime scene is collected, the temperature of the body is taken once more and found to be 78.5°F. This last temperature reading was taken exactly one hour after the first one. The next day the investigating detective asks the forensic expert, "What time did our victim die?" Assuming that the victim's body temperature was normal (98.6°F) prior to death, what does she tell the detective?

4. Population Problems. Consider a population p of field mice that grows at a rate proportional to the current population, so that $dp/dt = rp$.

(a) Find the rate constant r if the population doubles in 30 days.
(b) Find r if the population doubles in N days.

5. The field mouse population in Example 3 satisfies the differential equation

$$dp/dt = 0.5p - 450.$$

(a) Find the time at which the population becomes extinct if $p(0) = 850$.
(b) Find the time of extinction if $p(0) = p_0$, where $0 < p_0 < 900$.
(c) Find the initial population p_0 if the population is to become extinct in 1 year.

6. Radioactive Decay. Experiments show that a radioisotope decays at a rate negatively proportional to the amount of the isotope present.

(a) Use the following variables and parameters to write down and solve an initial value problem for the process of radioactive decay: t = time; $a(t)$ = amount of the radioisotope present at time t; a_0 = initial amount of radioisotope; r = decay rate, where $r > 0$.
(b) The **half-life**, $T_{1/2}$, of a radioisotope is the amount of time it takes for a quantity of the radioactive material to decay to one-half of its original amount. Find an expression for $T_{1/2}$ in terms of the decay rate r.

7. A radioactive material, such as the isotope thorium-234, disintegrates at a rate proportional to the amount currently present. If $Q(t)$ is the amount present at time t, then $dQ/dt = -rQ$, where $r > 0$ is the decay rate.

(a) If 100 milligrams (mg) of thorium-234 decays to 82.04 mg in 1 week, determine the decay rate r.
(b) Find an expression for the amount of thorium-234 present at any time t.
(c) Find the time required for the thorium-234 to decay to one-half its original amount.

8. Classical Mechanics. The differential equation for the velocity v of an object of mass m, restricted to vertical motion and subject only to the forces of gravity and air resistance, is

$$m\frac{dv}{dt} = -mg - \gamma v. \qquad \text{(i)}$$

In Eq. (i) we assume that the drag force, $-\gamma v$ where $\gamma > 0$ is a drag coefficient, is proportional to the velocity.

Acceleration due to gravity is denoted by g. Assume that the upward direction is positive.

(a) Show that the solution of Eq. (i) subject to the initial condition $v(0) = v_0$ is

$$v = \left(v_0 + \frac{mg}{\gamma}\right) e^{-\gamma t/m} - \frac{mg}{\gamma}.$$

(b) Sketch some integral curves, including the equilibrium solution, for Eq. (i). Explain the physical significance of the equilibrium solution.
(c) If a ball is initially thrown in the upward direction so that $v_0 > 0$, show that it reaches its maximum height when

$$t = t_{\max} = \frac{m}{\gamma} \ln\left(1 + \frac{\gamma v_0}{mg}\right).$$

(d) The terminal velocity of a baseball dropped from a high tower is measured to be 33 m/s. If the mass of the baseball is 145 grams (g) and $g = 9.8$ m/s^2, what is the value of γ?
(e) Using the values for m, g, and γ in part (d), what would be the maximum height attained for a baseball thrown upward with an initial velocity $v_0 = 30$ m/s from a height of 2 m above the ground?

9. For small, slowly falling objects, the assumption made in Eq. (i) of Problem 8 that the drag force is proportional to the velocity is a good one. For larger, more rapidly falling objects, it is more accurate to assume that the drag force is proportional to the square of the velocity.[2]

(a) Write a differential equation for the velocity of a falling object of mass m if the drag force is proportional to the square of the velocity. Assume that the upward direction is positive.
(b) Determine the limiting velocity after a long time.
(c) If $m = 0.025$ kilograms (kg), find the drag coefficient so that the limiting velocity is -35 m/s.

Mixing Problems. Many physical systems can be cast in the form of a mixing tank problem. Consider a tank containing a solution—a mixture of solute and solvent–such as salt dissolved in water. Assume that the solution at concentration $c_i(t)$ flows into the tank at a volume flow rate $r_i(t)$ and is simultaneously pumped out at the volume flow rate $r_o(t)$. If the solution in the tank is well mixed, then the concentration of the outflow is $Q(t)/V(t)$, where $Q(t)$ is the amount of solute at time t and $V(t)$ is the volume of solution in the tank. The differential equation that models the changing amount of solute in the tank is based on the principle of conservation of mass,

$$\underbrace{\frac{dQ}{dt}}_{\text{rate of change of } Q(t)} = \underbrace{c_i(t) r_i(t)}_{\text{rate in}} - \underbrace{\{Q(t)/V(t)\}\, r_o(t)}_{\text{rate out}}, \qquad \text{(i)}$$

where $V(t)$ also satisfies a mass conservation equation,

$$\frac{dV}{dt} = r_i(t) - r_o(t). \qquad \text{(ii)}$$

If the tank initially contains an amount of solute Q_0 in a volume of solution, V_0, then initial conditions for Eqs. (i) and (ii) are $Q(0) = Q_0$ and $V(0) = V_0$, respectively.

10. A tank initially contains 200 liters (L) of pure water. A solution containing 1 g/L enters the tank at a rate of 4 L/min, and the well-stirred solution leaves the tank at a rate of 5 L/min. Write initial value problems for the amount of salt in the tank and the amount of brine in the tank, at any time t.

11. A tank contains 100 gallons (gal) of water and 50 ounces (oz) of salt. Water containing a salt concentration of $\frac{1}{4}(1 + \frac{1}{2}\sin t)$ oz/gal flows into the tank at a rate of 2 gal/min, and the mixture flows out at the same rate. Write an initial value problem for the amount of salt in the tank at any time t.

12. A pond initially contains 1,000,000 gal of water and an unknown amount of an undesirable chemical. Water containing 0.01 g of this chemical per gallon flows into the pond at a rate of 300 gal/h. The mixture flows out at the same rate, so the amount of water in the pond remains constant. Assume that the chemical is uniformly distributed throughout the pond.

(a) Write a differential equation for the amount of chemical in the pond at any time.
(b) How much of the chemical will be in the pond after a very long time? Does this limiting amount depend on the amount that was present initially?

13. Pharmacokinetics. A simple model for the concentration $C(t)$ of a drug administered to a patient is based on the assumption that the rate of decrease of $C(t)$ is negatively proportional to the amount present in the system,

$$\frac{dC}{dt} = -kC,$$

where k is a rate constant that depends on the drug and its value can be found experimentally.

(a) Suppose that a dose administered at time $t = 0$ is rapidly distributed throughout the body, resulting in an initial concentration C_0 of the drug in the patient. Find $C(t)$, assuming the initial condition $C(0) = C_0$.
(b) Consider the case where doses of C_0 of the drug are given at equal time intervals T, that is, doses of C_0 are administered at times $t = 0, T, 2T, \ldots$. Denote by C_n the concentration immediately after the nth dose. Find an expression for the concentration C_2 immediately after the second dose.
(c) Find an expression for the concentration C_n immediately after the nth dose. What is $\lim_{n\to\infty} C_n$?

[2]See Lyle N. Long and Howard Weiss, "The Velocity Dependence of Aerodynamic Drag: A Primer for Mathematicians," *American Mathematical Monthly 106*, no. 2 (1999), pp. 127–135.

14. A certain drug is being administered intravenously to a hospital patient. Fluid containing 5 mg/cm^3 of the drug enters the patient's bloodstream at a rate of 100 cm^3/h. The drug is absorbed by body tissues or otherwise leaves the bloodstream at a rate proportional to the amount present, with a rate constant of 0.4 (h)$^{-1}$.

(a) Assuming that the drug is always uniformly distributed throughout the bloodstream, write a differential equation for the amount of the drug that is present in the bloodstream at any time.
(b) How much of the drug is present in the bloodstream after a long time?

Continuously Compounded Interest. The amount of money $P(t)$ in an interest bearing account in which the principal is compounded continuously at a rate r per annum and in which money is continuously added, or subtracted, at a rate of k dollars per annum satisfies the differential equation

$$\frac{dP}{dt} = rP + k. \tag{i}$$

The case $k < 0$ corresponds to paying off a loan, while $k > 0$ corresponds to accumulating wealth by the process of regular contributions to an interest bearing savings account.

15. Show that the solution to Eq. (i), subject to the initial condition $P(0) = P_0$, is

$$P = \left(P_0 + \frac{k}{r}\right)e^{rt} - \frac{k}{r}. \tag{ii}$$

Use Eq. (ii) in Problem 15 to solve Problems 16 and 17.

16. According to the International Institute of Social History (Amsterdam), the amount of money used to purchase Manhattan Island in 1626 is valued at \$1,050 in terms of today's dollars. If that amount were instead invested in an account that pays 4% per annum with continuous compounding, what would be the value of the investment in 2020? Compare with the case that interest is paid at 6% per annum.

17. How long will it take to pay off a student loan of \$20,000 if the interest paid on the principal is 5% and the student pays \$200 per month. What is the total amount of money repaid by the student?

18. Derive Eq. (ii) in Problem 15 from the discrete approximation to the change in the principal that occurs during the time interval $[t, t + \Delta t]$,

$$P(t + \Delta t) \cong P(t) + (r\Delta t)P(t) + k\Delta t,$$

assuming that $P(t)$ is continuously differentiable on $t \geq 0$. [*Hint:* Substitute $P(t + \Delta t) = P(t) + P'(t)\Delta t + (1/2)P''(\hat{t})(\Delta t)^2$), where $t < \hat{t} < t + \Delta t$, simplify, divide by Δt, and let $\Delta t \to 0$.]

Miscellaneous Modeling Problems

19. A spherical raindrop evaporates at a rate proportional to its surface area. Write a differential equation for the volume of the raindrop as a function of time.

20. Archimedes's *principle of buoyancy* states that an object submerged in a fluid is buoyed up by a force equal to the weight of the fluid displaced. An experimental, spherically shaped sonobuoy of radius 1/2 m with a mass m kg is dropped into the ocean with a velocity of 10 m/s when it hits the water. The sonobuoy experiences a drag force due to the water equal to one-half its velocity. Write down a differential equation describing the motion of the sonobuoy. Find values of m for which the sonobuoy will sink and calculate the corresponding terminal sink velocity of the sonobuoy. The density of seawater is $\rho_0 = 1.025$ kg/L.

1.2 Qualitative Methods: Phase Lines and Direction Fields

In Section 1.1 we were able to find solutions of the differential equations

$$\frac{du}{dt} = -k(u - T_0) \qquad \text{and} \qquad \frac{dp}{dt} = rp - k \tag{1}$$

by using a simple integration technique. Do not assume that this is always possible. Finding closed-form analytic solutions of differential equations can be difficult or impossible. Fortunately, it is possible to obtain information about the qualitative behavior of solutions by using elementary ideas from calculus and graphical methods; we consider two such methods in this section—phase line diagrams and direction fields.

Qualitative behavior refers to general properties of the differential equation and its solutions such as existence of equilibrium points, behavior of solutions near equilibrium points, and long-time behavior of solutions.[1] Qualitative analysis is important to the mathematical modeler because it can provide insight into even a very complicated model without having to find an exact solution or an approximation to an exact solution. It can show, often with only a small amount of effort, whether the equations are a plausible model of the phenomenon being studied. If not, what changes need to be made in the equations?

Autonomous Equations: Equilibrium Solutions and the Phase Line

A first order **autonomous** differential equation is an equation of the form

$$\frac{dy}{dt} = f(y). \tag{2}$$

The distinguishing feature of an autonomous equation is that the independent variable, in this case t, does not appear on the right side of the equation. For instance, the two equations appearing in (1) are autonomous. Other examples of autonomous equations are

$$p' = rp(1 - p/K), \qquad x' = \sin x, \qquad \text{and} \qquad y' = \sqrt{k^2/y - 1},$$

where r, K, and k are constants. However, the equations

$$u' + ku = kT_0 + kA \sin \omega t, \qquad x' = \sin(tx), \qquad \text{and} \qquad y' = -y + t$$

are not autonomous because the independent variable t does appear on the right side of each equation.

Equilibrium Solutions. The first step in a qualitative analysis of Eq. (2) is to find constant solutions of the equation. If $y = \phi(t) = c$ is a constant solution of Eq. (2), then $dy/dt = 0$. Therefore any constant solution must satisfy the algebraic equation

$$f(y) = 0. \tag{3}$$

These solutions are called **equilibrium solutions** of Eq. (2) because they correspond to no change or variation in the value of y as t increases or decreases. Equilibrium solutions are also referred to as **critical points**, **fixed points**, or **stationary points** of Eq. (2).

Equilibrium solutions, although simple, are usually important for understanding the behavior of other solutions of the differential equation. To obtain information about other solutions, we draw the graph of $f(y)$ versus y. Figure 1.2.1 shows a generic plot of $f(y)$, where the equilibrium points are $y = a$, b, and c. It is convenient to think of the variable y as the position of a particle whose motion along the horizontal axis is governed by Eq. (2). The corresponding velocity of the particle, dy/dt, is prescribed by Eq. (2).

At points where the velocity of the particle $dy/dt = f(y) > 0$, so that y is an increasing function of t, the particle moves to the right. This is indicated in Figure 1.2.1 by placing on the y-axis arrows that point to the right in the intervals $y < a$ and $b < y < c$ where $f(y) > 0$. At points where the velocity of the particle $dy/dt = f(y) < 0$, so that y is a decreasing function of t, the particle moves to the left. This is indicated in Figure 1.2.1 by placing on the y-axis arrows that point to the left in the intervals $a < y < b$ and $y > c$, where $f(y) < 0$.

[1] In addition, the qualitative properties of differential equations include results about existence and uniqueness of solutions, intervals of existence, and dependence of solutions on parameters and initial conditions. These issues will be addressed in Sections 2.4 and 2.5.

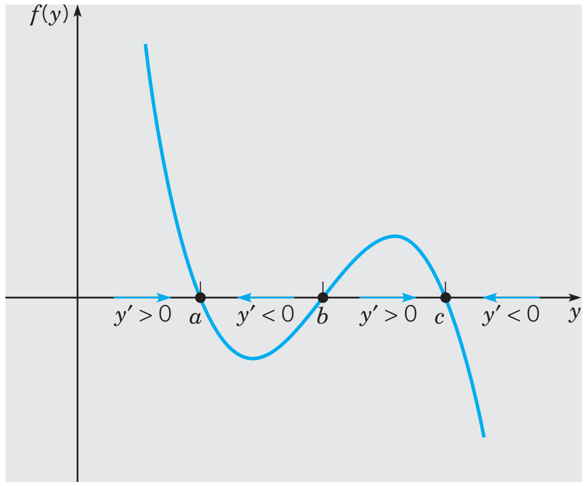


FIGURE 1.2.1 A generic graph of the right side of Eq. (2). The arrows on the y-axis indicate the direction in which y is changing [given by the sign of $y' = f(y)$] for each possible value of y. At the equilibrium points $y = a$, b, and c, $dy/dt = 0$.

The particle is stationary at the equilibrium points $y = a, b$, and c since $dy/dt = 0$ at each of those points.

The horizontal line in Figure 1.2.1 is referred to as the **phase line**, or the **one-dimensional phase portrait** of Eq. (2). The information contained in the phase line can be used to sketch the qualitatively correct integral curves of Eq. (2) by drawing it vertically just to the left of the ty-plane, as shown in Figure 1.2.2. We first draw the equilibrium solutions $y = a, b$ and c; then we draw a representative sampling of other curves that are increasing when $y < a$ and $b < y < c$ and decreasing when $a < y < b$ and $y > c$, as shown in Figure 1.2.2b.

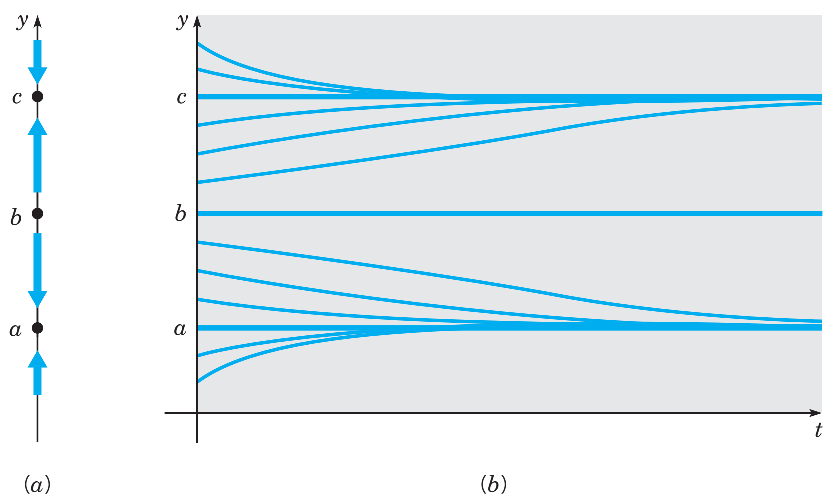


FIGURE 1.2.2 (a) The phase line. (b) Plots of y versus t.

Stability of Equilibrium Points. In the drawings of the phase line notice that arrows drawn on either side of the equilibrium point $y = a$ point toward $y = a$. Consequently, solution curves in Figure 1.2.2*b* that start sufficiently close to $y = a$ approach $y = a$ as $t \to \infty$. Similarly, arrows drawn on either side of $y = c$ in Figures 1.2.1 and 1.2.2*a* point toward $y = c$. It follows that solution curves that start sufficiently close to $y = c$ approach $y = c$ as $t \to \infty$, as shown in Figure 1.2.2*b*. The equilibrium points $y = a$ and $y = c$ are said to be **asymptotically stable**. On the other hand, arrows in the phase line that lie on either side of the equilibrium point $y = b$ point away from $y = b$. Correspondingly, solution curves that start near $y = b$ move away from $y = b$ as t increases. The equilibrium point $y = b$ is said to be **unstable**.

To facilitate our understanding of asymptotically stable and unstable equilibrium points, it is again useful to think of y as the position of a particle whose dynamics are governed by Eq. (2). A particle, perturbed slightly via some disturbance, from an asymptotically stable equilibrium point, will move back toward that point. However, a particle situated at an unstable equilibrium point, subjected to any disturbance, will move away from that point. *All real-world systems are subject to disturbances, most of which are unaccounted for in a mathematical model. Therefore, systems residing at unstable equilibrium points are not likely to be observed in the real world.*

EXAMPLE 1

Draw phase line diagrams for Eq. (2) of Section 1.1,

$$\frac{du}{dt} = -k(u - T_0), \qquad \text{where } k > 0, \tag{4}$$

and use it to discuss the behavior of all solutions as $t \to \infty$. Compare behaviors for two different values of k, $0 < k_1 < k_2$.

As shown in Figure 1.2.3*a*, the graph of $f(u) = -k_1(u - T_0)$ versus u is a straight line with slope $-k_1 < 0$ that intersects the phase line at $u = T_0$, the only equilibrium solution of Eq. (4). Since $u' > 0$ if $u < T_0$ and $u' < 0$ if $u > T_0$, all arrows on the phase line point toward $u = T_0$, which is therefore asymptotically stable. Consequently, any solution $u = \phi(t)$ of Eq. (4) satisfies

$$\lim_{t\to\infty} \phi(t) = T_0.$$

Equation (4) and Figure 1.2.3*a* also show that the absolute value of the instantaneous rate of heat exchange (as measured by $|u'|$) is an increasing function of the difference between the temperature of the object and the temperature of the surroundings,

$$|u'| = k_1|u - T_0|.$$

Thus the slope of any solution curve will be steeper at points far away from T_0 compared to points that are close to T_0. Furthermore the slope will approach zero as $|u - T_0| \to 0$. Solution curves consistent with these observations are shown in Figure 1.2.3*b*.

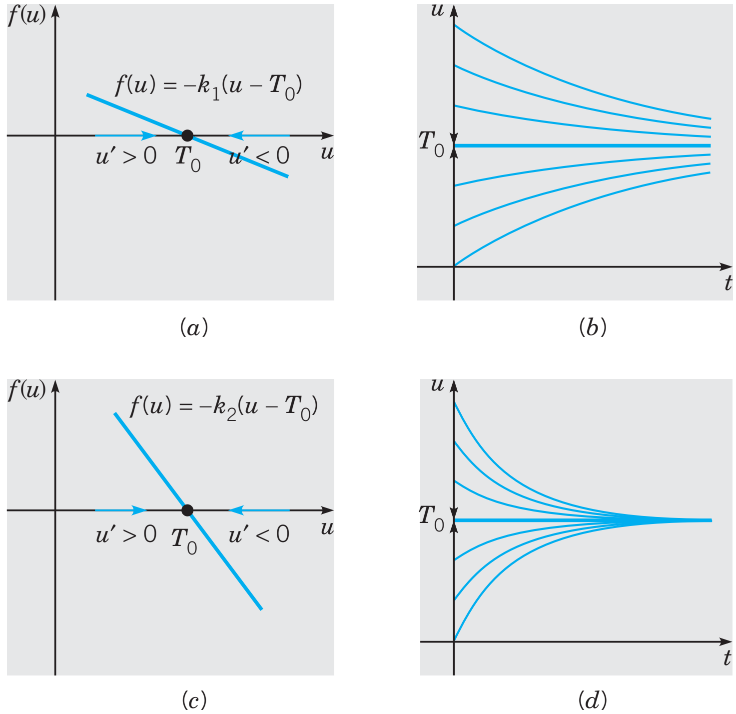


FIGURE 1.2.3 (*a*) and (*c*) Phase lines for $du/dt = -k(u - T_0)$, $k = k_1$ and k_2, where $k_1 < k_2$. The heavy blue arrows on the u-axis indicate the direction in which u is changing [given by the sign of $u'(t)$] for each possible value of u. For a given temperature difference $u - T_0$, the instantaneous rate of heat exchange depends on the slope $-k$ of the line. The parameter k is called the transmission coefficient. (*b*) and (*d*) Corresponding solutions of $du/dt = -k(u - T_0)$, where the phase line information in (*a*) and (*c*) is overlaid on the vertical axes. The rate of approach to equilibrium is governed by k. If k is small, the rate of heat exchange is slow. If k is large, the rate of heat exchange is rapid.

EXAMPLE 2

Draw a phase line diagram for the mouse population growth model, Eq. (15) of Section 1.1,

$$\frac{dp}{dt} = rp - a, \qquad \text{where } r, a > 0, \tag{5}$$

and use it to describe the behavior of all solutions as $t \to \infty$. Discuss implications of the model for the fruit farmer.

The only equilibrium solution of Eq. (5) is $p = a/r$. A plot of $f(p) = rp - a$ versus p in Figure 1.2.4*a* illustrates that $p' < 0$ when $p < a/r$, and $p' > 0$ when $p > a/r$. Thus the arrows on the p-axis point away from the equilibrium solution, which is unstable. Corresponding solution curves are shown in Figure 1.2.4*b*; note that the phase line diagram is overlaid on the p-axis.

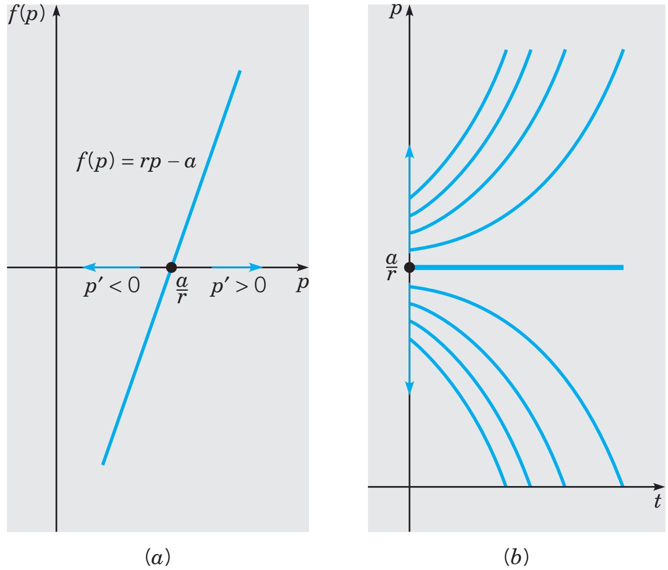


FIGURE 1.2.4 (a) The phase line for Eq. (5), $dp/dt = rp - a$, where $r, a > 0$. The slope r of the line corresponds to the growth rate of the mouse population. The direction of the arrows on the p-axis shows that the equilibrium solution $p = a/r$ is unstable. (b) Integral curves for Eq. (5).

Since the equilibrium solution is unstable, as time passes, an observer may see a mouse population either much larger or much smaller than the equilibrium population, but the equilibrium solution itself will not, in practice, be observed. Without the possible benefits of a more accurate and complex population model,[2] one inference that the fruit farmer might draw is that if he wants to control the mouse population, then he must install enough nesting boxes for the owls, thereby increasing the harvest rate a, to ensure that the mouse population $p(t)$ is always less than a/r. Thus a/r is a threshold value that should never be exceeded by $p(t)$ if the control strategy is to succeed.

This model also suggests a number of questions that the fruit farmer may wish to pursue, perhaps with assistance from a biologist who is knowledgeable about life cycles and habitats of field mice and owls:

- What is the growth rate of a field mouse population when there is an abundant food supply?
- How many mice per day does a barn owl consume?
- How do we estimate the size of the mouse population?
- Should we model the owl population?
- What will be a sustainable owl population if the mouse population drops to an economically acceptable level.

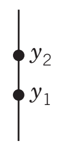
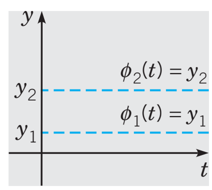
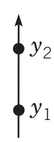
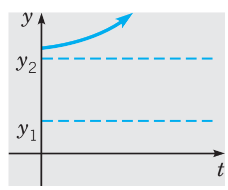
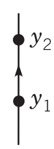
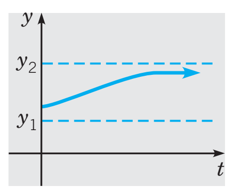
In each of the above examples, equilibrium solutions are important for understanding how other solutions of the given differential equation behave. An equilibrium solution may be thought of as a solution that serves as a reference to other, often nearby, solutions. An

[2]More elaborate population models appear in Sections 2.5 and 7.4.

asymptotically stable equilibrium solution is often referred to as an **attractor** or **sink**, since nearby solutions approach it as $t \to \infty$. On the other hand, an unstable equilibrium solution is referred to as a **repeller** or **source**.

The main steps for creating the phase line and a rough sketch of solution curves for a first-order autonomous differential equation are summarized in Table 1.2.1.

TABLE 1.2.1 Procedure for drawing phase lines and sketching solution curves for an autonomous equation.

	Illustration	
Step	**Phase Line**	**Solution Curves**
1. Find the equilibrium solutions of $dy/dt = f(y)$.	Solve $f(y) = 0$.	
2. Sketch the equilibrium solutions. These partition the phase line and ty-plane into disjoint regions.	Plot equilibrium solutions as points along a vertical line in increasing order as you move upward along the line. For instance, if $y_1 < y_2$ are equilibrium solutions, the phase line looks like 	Plot equilibrium solutions as dashed horizontal lines in the ty-plane. For instance, if $0 < y_1 < y_2$ are equilibrium solutions, the ty-plane looks like 
3. In each region, assess the sign of $f(y)$. (a) If $f(y) > 0$, then the solution curves passing through points in that region are increasing for all t, and either: (i) $\lim_{t\to\infty} y(t) = \infty$ if there is no larger equilibrium solution.	Affix arrowheads appropriately in each region. 	Sketch a representative solution curve in each region. 
(ii) $\lim_{t\to\infty} y(t) = y_2$ if y_2 is the next larger equilibrium solution.		

(continued)